Mohr-Coulomb Yielding

Theory: For the Mohr-Coulomb rheology, we can write the yielding relation as

\[ \sigma_s = \tan \varphi \sigma_n + C \]

where \( \sigma_s \) is the shear stress perpendicular to the fault plane, \( \sigma_n \) is the stress normal to the plane, \( \varphi \) is the angle of internal friction, and \( C \) is cohesion. Decomposing this into principal stresses gives us

\[ \sin(2\Theta)\theta/2 = \tan\varphi (\sigma_n + C)/2 + \cos(2\Theta)\theta/2 \]

where \( \Theta \) is the angle that the fault makes relative to the maximum shear stress. We then assume that the fault forms when the shear stress \( \sigma_s = 0 \) is a minimum. After a little algebra, this gives

\[ \Theta = +(\pi/4 + \varphi/2) \]

Numerical Shortening Experiment: We performed a simple shortening experiment as in Figure 1 for a variety of \( \varphi \). We solve the Stokes equation and look at the strain rate invariant to find incipient faults. This removes any of the confounding effects of strain weakening or time stepping.

Figure 2 shows a result for three different resolutions for \( \varphi = 45 \) degrees. Figure 3 shows a plot of the calculated vs expected values of the fault angles. This agreement gives us confidence in our implementation of Mohr Coulomb.

Magmatic Dikes

Theory: In the context of Stokes flow simulations, a useful approximation is to treat magmatic dikes as a region where material is created from thin air and flows out. We start by writing down the Stokes equations

\[ u_{ij,j} + \sigma_{ij,i} + p,i = f,i \]

where

\[ u = \text{velocity} \]
\[ p = \text{pressure} \]
\[ f = \text{body forces (e.g. gravity)} \]

We can introduce creation or destruction of material by adding a divergence term to the continuity equation

\[ \nabla \cdot u = D. \]

The question is then: how does this divergence term affect the dynamics of the simulation? To clarify this, consider an isoviscous case. In that case, the stress becomes

\[ \sigma_{ij} = \eta (u_{ij} + u_{ji}), \]

where

\[ \eta = \text{viscosity}. \]

Putting in the divergence term

\[ \eta (u_{ij} + D,i) + p,i = f,i \]

In the magma chamber, the viscosity is low, so we approximate cross derivatives as zero. Restricting temporarily to one dimension,

\[ u_{xx} = 0 \]
\[ \eta (u_{xx} + D) + p,x = f,x \]
\[ \eta D + p,x = f,x \]

Subtracting out the hydrostatic equilibrium implies

\[ p = 2nD + \text{constant} \]

So if there is a constant divergence inside the magma chamber and zero outside, this creates a pressure jump of \( 2nD \).

You can also go through the calculations for a sphere or ellipsoid and get the same result.