Abstract: We investigate the techniques needed to make a stable, robust method for solving large, 3D, highly nonlinear lithospheric problems. The Fix Causes Excess Compressibility

Unfortunately, when the method is applied to the lithosphere, serious problems arise. In our sandbox example, the pressure is dominated by the dynamic pressure, not the hydrostatic pressure. The dynamic pressure, in turn, is created by the motions arising from the velocities. Because of this link between the velocity and the pressure, the overall magnitude of the extra $C P$ term tends to be relatively small.

In lithospheric problems, on the other hand, the hydrostatic pressure can be several orders of magnitude larger than the dynamic pressure. Since the hydrostatic pressure is not linked to the velocity, it is easy for the extra term $C P$ to be far too large, even though $C$ itself is small and gets smaller as the resolution increases. Figure 3 shows how the dynamics of a simple extension model can get completely disrupted.

The Fix Causes Excess Compressibility

However, the nominal density profile has been subtracted away, so there is no need to counteract the hydrostatic pressure. If material piles up, the excess will not be subtracted away. That excess material then exerts a force that pushes the material down and material flows out as expected.

The converse is not true, though. If the material thins out, then there is no force on the bottom pushing the material up. We are still subtracting the nominal profile, so there should be a negative pressure resulting from the negative density above the simulation. To model this, we apply a stress to the top of the domain that is equal to the nominal pressure from the nominal density. As shown in Figure 6, that pulls up the material when it sinks too low.

The Fix Causes Excess Compressibility

As will be shown later, it is important to make a decent choice for the density profile, although it need not be perfect. For simple geometries, this is usually easy to do and results in a better solution, as shown in Figure 4.

The Fix Causes Excess Compressibility


Simple Q1-Q1 Fix

Dohrmann and Bochev (2000) introduced a relatively simple scheme using Q1-Q1 elements and a stabilization term that looks like a compressibility. Specifically, the continuity equation is modified to become

$$\nabla \cdot \mathbf{v} + C P = 0$$

where

$$C \equiv \frac{1}{\rho} \left( \int_{\Omega} \psi(x) \nabla \cdot \mathbf{v}^T(x) d\Omega_x - \int_{\Omega} \nabla \cdot \mathbf{v}^T d\Omega_x \right)$$

is the compressibility associated with the nominal density profile. When computing the true pressure (e.g., for yielding calculations or visualization) we have to remember to add it back in.

The Fix Causes Excess Compressibility

As the crust thins, the heavy asthenospheric material flows up to fill in the void left by the weak zone. However, this triggers an interesting instability. The heavy material creates an excess pressure. This pressure makes the $C P$ term larger. There is then a coupling which causes even more pressure, which increases $C P$, and so on.

The Fix Causes Excess Compressibility

The fix for this is to change the nominal density profile so that it is all made up of the heavier material. The mechanism behind the instability and the workaround is not completely clear.

Boundary Conditions

A reasonable scheme for subtracting out the hydrostatic pressure is to subtract the profile of the asthenosphere, mantle, and crust. We can not worry about the pressure in space, because that would cause dynamics that are not accounted for. So the weak zone will not be exactly subtracted, but as the simulation shows, this does not seem to cause any problems.

Conclusions

The techniques presented here make it possible to simulate highly nonlinear Stokes flow problems with the relatively simple Q1-Q1 finite elements. This allows us to use particle methods in a straightforward manner. That, in turn, lets us capture strain history, allowing us to follow simulations with large deformations with no diffusion.

With that said, the introduction of the nominal density profile adds as many problems as it solves. Getting all of the parameters just right can be an arduous exercise.

Figure 8: Simulation of a tabletop model with multiple layers. As the material extends, the weak zones are more quickly allowing the dense asthenosphere to fill the void. If the nominal density profile is set up to reflect the differing densities in the crust, mantle, and asthenosphere, then an instability occurs since the asthenosphere is filling in the void. If, instead, the profile assumes that the dense asthenosphere is the only material, then the simulation succeeds. Model courtesy of John Sheehan.